

# CONTINUOUS TIME PORTFOLIO CHOICE UNDER MONOTONE PREFERENCES WITH QUADRATIC PENALTY - STOCHASTIC FACTOR CASE

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**ABSTRACT.** We consider an incomplete market with an non-tradable stochastic factor and an investment problem with optimality criterion based on a functional which is a modification of a monotone mean-variance preferences. We formulate it as a stochastic differential game problem and use Hamilton-Jacobi-Bellman-Isaacs equations to derive the optimal investment strategy and the value function. Finally, we show that our solution coincides with the solution to classical Markowitz problem with risk aversion coefficient which is dependent on stochastic factor.

## 1. INTRODUCTION

Since Markowitz published his famous paper, a quadratic optimization problem has gained a lot of attraction in asset allocation and active portfolio management. Nevertheless, it is well known that mean-variance functional is not monotone. For this reason Maccheroni et al. [13] created a new class of monotone preferences that coincide with mean-variance preferences on their domain of monotonicity, but differ where mean-variance preferences fail to be monotone. Moreover, they showed the functional associated with this class of preferences is the best approximation of the mean-variance functional among those which are monotonic. For more details about the monotone mean-variance preferences and its other advantages over mean-variance preferences we refer to [13].

In this paper we assume that an investor has access to the market, where he can freely buy and sale a riskless bond and a risky asset whose price is a diffusion with dynamics affected by a correlated non-tradable (but observable) stochastic factor. The purpose is to describe an optimal financial strategy which an investor can follow in order to maximize his performance criterion which is a modification of the monotone mean-variance functional. Let us recall that Maccheroni et al. [13] introduce a functional given by

$$(1.1) \quad X \rightarrow \inf_{Q \in \mathcal{Q}} \mathbb{E}^Q \left[ X + \frac{1}{2\theta} C(Q|P) \right], \quad \theta > 0,$$

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2010 *Mathematics Subject Classification.* 91G10; 91A15; 91A23; 93E20.

*Key words and phrases.* Stochastic factor, HJBI equations, stochastic games, convex risk measures.

where  $\theta$  is a risk aversion coefficient,  $\mathcal{Q}$  is a class of all probability measures,  $P$  is a given probability measure and

$$C(Q|P) = \begin{cases} \mathbb{E}^P \left[ \left( \frac{dQ}{dP} \right)^2 \right] - 1, & \text{if } Q \ll P, \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that this form of  $C(Q|P)$  is sometimes called the Gini concentration index.

Nevertheless, due to technical difficulties, we consider in (1.1) only the set  $\mathcal{Q}$  which consists of all absolutely continuous probability measures which have square integrable Radon-Nikodym derivative of the form (2.2). This modification of monotone mean-variance functional is still monotone and additional motivation to consider such type of performance criterion lies in the fact that

$$\Lambda(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q \left[ -X - \frac{1}{2\theta} \frac{dQ}{dP} \right], \quad \theta > 0,$$

satisfies the following three axioms:

*Convexity:* If  $\alpha \in (0, 1)$ , then  $\Lambda(\alpha X + (1 - \alpha)Y) \leq \alpha \Lambda(X) + (1 - \alpha)\Lambda(Y)$ .

*Monotonicity:* If  $X \leq Y$ , then  $\Lambda(Y) \leq \Lambda(X)$ .

*Translation invariance:* If  $\beta \in \mathbb{R}$ , then  $\Lambda(X + \beta) = \Lambda(X) - \beta$ .

Namely,  $\Lambda(X)$  is a convex risk measure (see Föllmer and Schied [8] or Frittelli and Rosazza-Gianin [9]) and minimization of  $\Lambda(X)$  might be interpreted as searching for risk minimizing portfolio. Note that, in this case the function  $C(Q|P)$  is called a penalty function.

The problem of looking for risk minimizing portfolio with various modifications of performance criterion (1.1) was considered by many authors. For example Mataramvura and Øksendal [15] studied this issue in jump diffusion setting with general penalty function

$$C(Q|P) = \zeta_0 \left( \frac{dQ}{dP} \right).$$

The same kind of problem was examined by Elliott and Siu in a regime switching market in [4] and [5] and in an insurer optimal reinsurance problem in [3]. Risk based portfolio problems are also useful to determine prices of derivatives in incomplete markets. Namely, one possibility is to determine the value by considering so called *risk indifference price*. For more information about indifference price in a jump diffusion market see Øksendal and Sulem [17], while for stochastic factor model it is worth to read Elliott and Siu [6]. It merits mentioning that in many papers utility function is used to take into account non-linear form of investors satisfaction in functional (1.1). This approach is called *robust utility portfolio optimization* and was taken up in Hernández and Schied [11] (stochastic factor model), Øksendal and Sulem [16] (jump-diffusion risk), Bordgioni et al. [2] (more general semimartingale setting).

All of the aforementioned papers examine the problem without presented any detailed solution for a specific choice of  $C(Q|P)$  or consider one specific example using entropic

penalty function

$$C(Q|P) = \frac{dQ}{dP} \ln \left( \frac{dQ}{dP} \right).$$

Due to our knowledge the penalty

$$C(Q|P) = \mathbb{E}^P \left[ \left( \frac{dQ}{dP} \right)^2 \right] - 1$$

has never been studied in detail in a dynamic optimization framework.

The problem of maximizing (1.1) is a max-min problem, hence it naturally forms a stochastic differential game. In the literature there are two main approaches in determining the solution to such games. First of them uses maximum principle and Backward Stochastic Differential Equations (BSDE), while the second one is based on dynamic programming principle and Hamilton-Jacobi-Bellman equations (Hamilton-Jacobi-Bellman-Isaacs (HJBI) for differential games). The most suitable for us is the latter. In our case associated HJBI equation can be simplified to a linear form by applying some transformations. As a by-product we obtain a formula for the optimal strategy. To complete the reasoning it is sufficient to use a proper version of the verification theorem.

The paper is organized as follows. In Section 2 we describe the set up of the problem. We formulate the verification theorem and derive the HJBI equation. In Section 3 we transform our equation to linear form and prove useful properties of its solution. In Section 4 we formulate the main theorem and compare the result with solution to classical mean-variance optimization problem.

## 2. GENERAL MODEL DESCRIPTION

Let  $(\mathcal{F}_t, 0 \leq t \leq T)$  be a filtration (possibly enlarged to satisfy usual assumptions) generated by two independent Brownian motions  $(W_t^1, 0 \leq t \leq T)$ ,  $(W_t^2, 0 \leq t \leq T)$  defined on  $(\Omega, \mathcal{F}, P)$ . Suppose that an investor has access to the market, where he can freely buy and sale two primitive securities: a bond  $(B_t, 0 \leq t \leq T)$  and a share  $(S_t, 0 \leq t \leq T)$ . We assume that the price of the share is modulated by one non-tradable (but observable) factor  $(Z_t, 0 \leq t \leq T)$ . This factor can represent an additional source of uncertainty such as stochastic volatility, varying economic conditions or non-financial risk (for instance weather risk). Processes mentioned above are solutions of the following system of stochastic differential equations

$$(2.1) \quad \begin{cases} dB_t = rB_t dt, \\ dS_t = \mu(Z_t)S_t dt + \sigma(Z_t)S_t dW_t^1, \\ dZ_t = a(Z_t)dt + b(Z_t)(\rho dW_t^1 + \bar{\rho} dW_t^2), \quad Z_s = z, \end{cases}$$

where the coefficients  $\mu, \sigma, a, b$  are continuous functions and they are assumed to satisfy all the required regularity conditions, in order to guarantee that the unique strong solution to (2.1) exists. The interest rate  $r > 0$  is constant,  $\rho \in [-1, 1]$  is a correlation coefficient and  $\bar{\rho} := \sqrt{1 - \rho^2}$ . The assumption of time-independent coefficients is for notational convenience only and can easily be relaxed.

In this paper we assume that the probability measure is not precisely known and the agent knows only a class of possible measures. To construct investors objective function, following Hernández and Schied [11], we will consider the class

$$(2.2) \quad \mathcal{Q} := \left\{ Q \sim P : \frac{dQ}{dP} = \mathcal{E} \left( \int \eta_{t,1} dW_t^1 + \eta_{t,2} dW_t^2 \right)_T, \quad (\eta_1, \eta_2) \in \mathcal{M} \right\},$$

where  $\mathcal{E}(\cdot)_t$  denotes the Doleans-Dade exponential and  $\mathcal{M}$  is the set of all progressively measurable processes  $\eta = (\eta_1, \eta_2)$  such that

$$\mathbb{E} \left( \frac{dQ^\eta}{dP} \right)^2 < +\infty \quad \text{and} \quad \mathbb{E} \left( \frac{dQ^\eta}{dP} \right) = 1,$$

where  $Q^\eta$  denotes the measure determined by  $\eta \in \mathcal{M}$ . It means we have additional family of stochastic processes  $(Y_t^\eta, 0 \leq t \leq T)$  which is given by the stochastic differential equation

$$dY_t^\eta = \eta_{t,1} Y_t^\eta dW_t^1 + \eta_{t,2} Y_t^\eta dW_t^2, \quad Y_s^\eta = y.$$

Moreover, notice that

$$Y_T^\eta = y \frac{dQ^\eta}{dP}.$$

Now, let us define  $(\bar{X}_t^{\bar{\pi}}, 0 \leq t \leq T)$  as the investors wealth process with the following dynamics

$$d\bar{X}_t^{\bar{\pi}} = (r\bar{X}_t^{\bar{\pi}} + \bar{\pi}_t(\mu(Z_t) - r))dt + \bar{\pi}_t\sigma(Z_t)dW_t^1, \quad \bar{X}_s^{\bar{\pi}} = \bar{x},$$

where  $\bar{x}$  denotes a current wealth of the investor, whereas a control  $\bar{\pi}$  we can interpret as a part of the wealth invested in  $S_t$ . Note that  $\bar{\pi}$  as well as the portfolio wealth  $\bar{X}_T^{\bar{\pi}}$  are allowed to be negative. In this work it is convenient for us to introduce  $T$ -forward values of  $\bar{\pi}$  and  $\bar{X}_t^{\bar{\pi}}$ . Namely, let

$$\pi_t := e^{r(T-t)}\bar{\pi}_t, \quad X_t^\pi := e^{r(T-t)}\bar{X}_t^{\bar{\pi}},$$

then the dynamics of the wealth process can be rewritten as

$$(2.3) \quad dX_t^\pi = \pi_t (\mu(Z_t) - r) dt + \pi_t \sigma(Z_t) dW_t^1.$$

**Definition 2.1.** A control (or strategy)  $\pi = (\pi_s, t \leq s \leq T)$  is admissible on the time interval  $[t, T]$ , written  $\pi \in \mathcal{A}_t$ , if it satisfies the following assumptions:

- (i)  $\pi$  is progressively measurable;
- (ii) unique solution to (2.3) exists and

$$\mathbb{E}_{x,t}^\eta \left[ \sup_{t \leq s \leq T} |X_s^\pi| \right] < +\infty \quad \text{for all } \eta \in \mathcal{M},$$

where  $\mathbb{E}^\eta$  denotes the expectation with respect to measure  $Q^\eta$ .

**Formulation of the problem.** As announced we consider Maccheroni type objective function

$$J^{\pi,\eta}(x, y, z, t) := \mathbb{E}_{x,y,z,t}^{\eta} [-X_T^{\pi} - Y_T^{\eta}].$$

The investors aim is to

$$(2.4) \quad \text{minimize} \quad \sup_{\eta \in \mathcal{M}} J^{\pi,\eta}(x, y, z, t)$$

over a class of admissible strategies  $\mathcal{A}_t$ .

The problem (2.4) is considered as a zero-sum stochastic differential game problem. Measure  $Q^{\eta}$  is the control of player number 1 (the „market”), while the portfolio  $\pi$  is the control of player number 2 (the „investor”). We are looking for a saddle point  $(\pi^*, \eta^*) \in \mathcal{A}_t \times \mathcal{M}$  and a value function  $V(x, y, z, t)$  such that

$$J^{\pi^*,\eta}(x, y, z, t) \leq J^{\pi^*,\eta^*}(x, y, z, t) \leq J^{\pi,\eta^*}(x, y, z, t),$$

and

$$V(x, y, z, t) = J^{\pi^*,\eta^*}(x, y, z, t).$$

For more information about differential games we refer to Fleming and Soner [7] and references therein.

**HJBI equations and the verification theorem.** The investment problem stated in the previous section can be solved by applying stochastic control theory. In this section we establish a link between Hamilton-Jacobi-Bellman-Isaacs equations and a saddle point of our initial problem.

Let us remind, that

$$(2.5) \quad \begin{cases} dX_t^{\pi} = \pi_t(\mu(Z_t) - r)dt + \pi_t\sigma(Z_t)dW_t^1, \\ dY_t^{\eta} = \eta_{t,1}Y_t^{\eta}dW_t^1 + \eta_{t,2}Y_t^{\eta}dW_t^2, \\ dZ_t = a(Z_t)dt + b(Z_t)(\rho dW_t^1 + \bar{\rho}dW_t^2). \end{cases}$$

It is convenient to consider  $Q^{\eta}$ -dynamics of system (2.5). After applying the Girsanov transformation, we have

$$(2.6) \quad \begin{cases} dX_t^{\pi} = \pi_t(\mu(Z_t) - r + \sigma(Z_t)\eta_{t1})dt + \pi_t\sigma(Z_t)dW_t^{\eta_1}, \\ dY_t^{\eta} = (\eta_{t1}^2 + \eta_{t2}^2)Y_t^{\eta}dt + \eta_{t1}Y_t^{\eta}dW_t^{\eta_1} + \eta_{t2}Y_t^{\eta}dW_t^{\eta_2}, \\ dZ_t = (a(Z_t) + b(Z_t)\rho\eta_{t1} + b(Z_t)\bar{\rho}\eta_{t2})dt + b(Z_t)(\rho dW_t^{\eta_1} + \bar{\rho}dW_t^{\eta_2}), \end{cases}$$

where  $(W_t^{\eta_1}, W_t^{\eta_2})$  are  $Q^{\eta}$ -Brownian motions defined as

$$\begin{cases} dW_t^{\eta_1} = dW_t^1 - \eta_{t1}dt, \\ dW_t^{\eta_2} = dW_t^2 - \eta_{t2}dt. \end{cases}$$

Let  $\mathcal{L}^{\pi, \eta}$  be the differential operator given by

$$\begin{aligned} \mathcal{L}^{\pi, \eta} V(x, y, z, t) &:= V_t + \pi(\mu(z) - r + \sigma(z)\eta_1)V_x + (\eta_1^2 + \eta_2^2)yV_y \\ &\quad + (a(z) + b(z)\rho\eta_1 + b(z)\bar{\rho}\eta_2)V_z + \frac{1}{2}\pi^2\sigma^2(z)V_{xx} \\ &\quad + \frac{1}{2}(\eta_1^2 + \eta_2^2)y^2V_{yy} + \frac{1}{2}b^2(z)V_{zz} + \pi\sigma(z)\eta_1yV_{xy} \\ &\quad + \pi\sigma(z)b(z)\rho V_{xz} + b(z)(\rho\eta_1 + \bar{\rho}\eta_2)yV_{yz}. \end{aligned}$$

We can now formulate the Verification Theorem. The proof of this theorem is very similar to the proof of analogous theorem from Mataramvura and Øksendal [15] or Zawisza [20] and [21], so in this paper we give only a sketch.

**Theorem 2.2** (Verification Theorem). *Suppose there exists a function*

$$V \in \mathcal{C}^{2,2,2,1}(\mathbb{R} \times (0, +\infty) \times \mathbb{R} \times [0, T)) \cap \mathcal{C}(\mathbb{R} \times [0, +\infty) \times \mathbb{R} \times [0, T])$$

*and a Markov control*

$$(\pi^*(x, y, z, t), \eta^*(x, y, z, t)) \in \mathcal{A}_t \times \mathcal{M},$$

*such that*

$$(2.7) \quad \mathcal{L}^{\pi^*(x, y, z, t), \eta^*(x, y, z, t)} V(x, y, z, t) \leq 0,$$

$$(2.8) \quad \mathcal{L}^{\pi, \eta^*(x, y, z, t)} V(x, y, z, t) \geq 0,$$

$$(2.9) \quad \mathcal{L}^{\pi^*(x, y, z, t), \eta^*(x, y, z, t)} V(x, y, z, t) = 0,$$

$$(2.10) \quad V(x, y, z, T) = -x - y$$

$$\text{for all } \eta \in \mathbb{R}^2, \pi \in \mathbb{R}, (x, y, z, t) \in \mathbb{R} \times (0, +\infty) \times \mathbb{R} \times [0, T),$$

*and*

$$(2.11) \quad \mathbb{E}_{x, y, z, t}^{\eta} \left[ \sup_{t \leq s \leq T} |V(X_s^{\pi}, Y_s, Z_s, s)| \right] < +\infty$$

$$\text{for all } (x, y, z, t) \in \mathbb{R} \times [0, +\infty) \times \mathbb{R} \times [0, T], \pi \in \mathcal{A}_t, \eta \in \mathcal{M}.$$

*Then*

$$J^{\pi^*, \eta}(x, y, z, t) \leq V(x, y, z, t) \leq J^{\pi, \eta^*}(x, y, z, t)$$

$$\text{for all } \pi \in \mathcal{A}_t, \eta \in \mathcal{M},$$

*and*

$$V(x, y, z, t) = J^{\pi^*, \eta^*}(x, y, z, t).$$

*Proof.* Fix  $(x, y, z, t) \in \mathbb{R} \times (0, +\infty) \times \mathbb{R} \times [0, T)$ , choose any  $\eta \in \mathcal{M}$  and consider system of equations (2.6) with  $\pi_t^* = \pi^*(X_t, Y_t, Z_t, t)$ . Then, if we apply the Itô formula to (2.6) and the function  $V$ , we get

$$\mathbb{E}_{x, y, z, t}^{\eta} \left[ V \left( X_{T_n^{\varepsilon} \wedge (T - \varepsilon)}^{\pi^*}, Y_{T_n^{\varepsilon} \wedge (T - \varepsilon)}^{\eta}, Z_{T_n^{\varepsilon} \wedge (T - \varepsilon)}, T_n^{\varepsilon} \wedge (T - \varepsilon) \right) \right] = V(x, y, z, t)$$

$$+\mathbb{E}_{x,y,z,t}^{\eta} \left[ \int_t^{T_n^{\varepsilon} \wedge (T-\varepsilon)} \mathcal{L}^{\pi_s^*, \eta_s} V(X_s^{\pi^*}, Y_s^{\eta}, Z_s, s) ds \right] + \mathbb{E}_{x,y,z,t}^{\eta} \left[ \int_t^{T_n^{\varepsilon} \wedge (T-\varepsilon)} M_s^{\varepsilon} dW_s^{\eta} \right],$$

where  $(T_n^{\varepsilon}, n = 1, 2, \dots)$  is a localizing sequence of stopping times such that  $(T_n^{\varepsilon} \uparrow T)$  and

$$\mathbb{E}_{x,y,z,t}^{\eta} \left[ \int_t^{T_n^{\varepsilon} \wedge (T-\varepsilon)} M_s^{\varepsilon} dW_s^{\eta} \right] = 0.$$

Using (2.7) yields

$$\mathbb{E}_{x,y,z,t}^{\eta} \left[ V \left( X_{T_n^{\varepsilon} \wedge (T-\varepsilon)}^{\pi^*}, Y_{T_n^{\varepsilon} \wedge (T-\varepsilon)}^{\eta}, Z_{T_n^{\varepsilon} \wedge (T-\varepsilon)}, T_n^{\varepsilon} \wedge (T-\varepsilon) \right) \right] \geq V(x, y, z, t).$$

Since (2.11) holds, we can apply the dominated convergence theorem. Letting  $n \rightarrow +\infty$ ,  $\varepsilon \rightarrow 0$  and using (2.10) we obtain

$$J^{\pi^*, \eta}(x, y, z, t) \geq V(x, y, z, t).$$

If we replace  $\eta$  by  $\eta^*$  and apply (2.9) we obtain

$$J^{\pi^*, \eta^*}(x, y, z, t) = V(x, y, z, t).$$

Next we choose any  $\pi \in \mathcal{A}_t$  and apply the Itô formula to system of equations (2.6) with  $\eta_t^* = \eta^*(X_t, Y_t, Z_t, t)$ . Repeating the method presented above and using (2.8) we get

$$J^{\pi, \eta^*}(x, y, z, t) \leq V(x, y, z, t).$$

□

Let us point out that conditions (2.7) - (2.10) hold if the upper and the lower Hamilton-Jacobi-Bellman-Isaacs equations are satisfied:

$$\begin{aligned} \min_{\pi \in \mathbb{R}} \max_{\eta \in \mathbb{R}^2} \mathcal{L}^{\pi, \eta} V(x, y, z, t) &= \max_{\eta \in \mathbb{R}^2} \min_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta} V(x, y, z, t) = 0, \\ V(x, y, z, T) &= -x - y. \end{aligned}$$

Once we verify that one of these equation has a unique solution  $V$ , it is also necessary to prove that  $V$  is also a solution to the second one. For more details see Lemma 2.4.

**Solution to the minimax problem.** To find the saddle point it is more convenient for us to use the upper Hamilton-Jacobi-Bellman-Isaacs equation

$$(2.12) \quad \min_{\pi \in \mathbb{R}} \max_{\eta \in \mathbb{R}^2} \mathcal{L}^{\pi, \eta} V(x, y, z, t) = 0,$$

i.e.

$$\begin{aligned}
& V_t + a(z)V_z + \frac{1}{2}b^2(z)V_{zz} \\
& + \min_{\pi \in \mathbb{R}} \max_{(\eta_1, \eta_2) \in \mathbb{R}^2} \left\{ \pi(\mu(z) - r + \sigma(z)\eta_1)V_x + (\eta_1^2 + \eta_2^2)yV_y \right. \\
& + b(z)(\rho\eta_1 + \bar{\rho}\eta_2)V_z + \frac{1}{2}\pi^2\sigma^2(z)V_{xx} + \frac{1}{2}(\eta_1^2 + \eta_2^2)y^2V_{yy} \\
& \left. + \pi\sigma(z)\eta_1yV_{xy} + \pi\sigma(z)b(z)\rho V_{xz} + b(z)(\rho\eta_1 + \bar{\rho}\eta_2)yV_{yz} \right\} = 0.
\end{aligned}$$

We expect  $V(x, y, z, t)$  to be of the form

$$V(x, y, z, t) = -x + G(z, t)y, \quad \text{where} \quad G(z, T) = -1.$$

Then we have

$$\begin{aligned}
& yG_t + a(z)yG_z + \frac{1}{2}b^2(z)yG_{zz} + \min_{\pi \in \mathbb{R}} \max_{(\eta_1, \eta_2) \in \mathbb{R}^2} \left\{ -\pi(\mu(z) - r + \sigma(z)\eta_1) \right. \\
& \left. + (\eta_1^2 + \eta_2^2)yG + 2b(z)(\rho\eta_1 + \bar{\rho}\eta_2)yG_z \right\} = 0.
\end{aligned}$$

The maximum over  $(\eta_1, \eta_2)$  is attained at  $(\eta_1^*, \eta_2^*)$ , where

$$\begin{aligned}
\eta_1^*(\pi) &= \frac{\sigma(z)}{2yG(z, t)}\pi - \rho b(z)\frac{G_z(z, t)}{G(z, t)}, \\
\eta_2^* &= -\bar{\rho}b(z)\frac{G_z(z, t)}{G(z, t)}.
\end{aligned}$$

For  $(\eta_1^*, \eta_2^*)$  our equation is of the form

$$\begin{aligned}
& yG_t + a(z)yG_z + \frac{1}{2}b^2(z)yG_{zz} + \min_{\pi \in \mathbb{R}} \left\{ -\pi(\mu(z) - r + \sigma(z)\eta_1^*(\pi)) \right. \\
& \left. + ((\eta_1^*(\pi))^2 + (\eta_2^*)^2)yG + 2b(z)(\rho\eta_1^*(\pi) + \bar{\rho}\eta_2^*)yG_z \right\} = 0.
\end{aligned} \tag{2.13}$$

The minimum over  $\pi$  is attained at

$$\pi^* = -2yG(z, t) \left[ \frac{\mu(z) - r}{\sigma^2(z)} - \frac{\rho b(z)}{\sigma(z)} \frac{G_z(z, t)}{G(z, t)} \right]. \tag{2.14}$$

It is worth to notice here that

$$\eta_1^*(\pi^*) = -\frac{\mu(z) - r}{\sigma(z)}, \tag{2.15}$$

so the saddle point candidate

$$(\pi^*, (\eta_1^*(\pi^*), \eta_2^*)) \tag{2.16}$$



looks as follows

$$\begin{aligned}\pi^* &= -2yG(z, t) \left[ \frac{\lambda(z)}{\sigma(z)} - \frac{\rho b(z)}{\sigma(z)} \frac{G_z(z, t)}{G(z, t)} \right], \\ \eta_1^*(\pi^*) &= -\lambda(z), \\ \eta_2^* &= -\bar{\rho}b(z) \frac{G_z(z, t)}{G(z, t)},\end{aligned}$$

where

$$\lambda(z) = \frac{\mu(z) - r}{\sigma(z)}.$$

Now we substitute (2.14) into (2.13) and after dividing by  $y$  we get the final equation of the form

$$(2.17) \quad G_t + (a(z) - 2\rho b(z)\lambda(z)) G_z + \frac{1}{2}b^2(z)G_{zz} - \bar{\rho}^2b^2(z)\frac{G_z^2}{G} + \lambda^2(z)G = 0.$$

**Remark 2.3.** In the next section we rewrite equation (2.17) with condition  $G(z, T) = -1$  to linear form and we give a set of assumptions to ensure existence of a unique solution.

Now we are ready to prove the following two lemmas.

**Lemma 2.4.** *If there exists function  $V$  which is a solution to (2.12), then it is also the solution to lower Hamilton-Jacobi-Bellman-Isaacs equation*

$$\max_{\eta \in \mathbb{R}^2} \min_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta} V(x, y, z, t) = 0.$$

*Proof.* We already know that

$$\max_{\eta \in \mathbb{R}^2} \min_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta} V(x, y, z, t) \leq \min_{\pi \in \mathbb{R}} \max_{\eta \in \mathbb{R}^2} \mathcal{L}^{\pi, \eta} V(x, y, z, t),$$

hence

$$\max_{\eta \in \mathbb{R}^2} \min_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta} V(x, y, z, t) \leq 0.$$

In addition we always have

$$\max_{\eta \in \mathbb{R}^2} \min_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta} V(x, y, z, t) \geq \min_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta} V(x, y, z, t), \quad \forall \eta \in \mathbb{R}^2.$$

Using (2.13) and (2.15) it is easy to verify that

$$\min_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta^*(\pi^*)} V(x, y, z, t) = 0,$$

so

$$\max_{\eta \in \mathbb{R}^2} \min_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta} V(x, y, z, t) \geq 0.$$

□

The second lemma will be helpful in Section 4 to prove the main result and establish the similarities between our paper and mean-variance optimization methods.

**Lemma 2.5.** *Suppose that initial conditions  $x_0, y_0, z_0, t_0$  are fixed,  $G$  is a solution to equation (2.17) and  $(\pi^*, (\eta_1^*(\pi^*), \eta_2^*)) \in \mathcal{A}_t \times \mathcal{M}$  is given by (2.16). Then*

$$2Y_t^{\eta^*} G(Z_t, t) = X_t^{\pi^*} - x_0 + 2y_0 G(z_0, t_0), \quad \forall t \in [t_0, T].$$

*Proof.* It is sufficient to prove only that

$$dX_t^{\pi^*} = d\left(2Y_t^{\eta^*} G(Z_t, t)\right).$$

First of all, note that for saddle point given by (2.16) system of equations (2.5) is of the form

$$(2.18) \quad \begin{aligned} dX_t^{\pi^*} = & -2Y_t^{\eta^*} G(Z_t, t) \left[ \lambda^2(Z_t) - \rho b(Z_t) \lambda(Z_t) \frac{G_z(Z_t, t)}{G(Z_t, t)} \right] dt \\ & - 2Y_t^{\eta^*} G(Z_t, t) \left[ \lambda(Z_t) - \rho b(Z_t) \frac{G_z(Z_t, t)}{G(Z_t, t)} \right] dW_t^1 \end{aligned}$$

and

$$dY_t^{\eta^*} = -\lambda(Z_t) Y_t^{\eta^*} dW_t^1 - \bar{\rho} b(Z_t) \frac{G_z(Z_t, t)}{G(Z_t, t)} Y_t^{\eta^*} dW_t^2.$$

Using (2.17) we can verify that

$$\begin{aligned} dG(Z_t, t) = & \left[ 2\rho b(Z_t) \lambda(Z_t) G_z(Z_t, t) + \bar{\rho}^2 b^2(Z_t) \frac{G_z^2(Z_t, t)}{G(Z_t, t)} - \lambda^2(Z_t) G(Z_t, t) \right] dt \\ & + G_z(Z_t, t) b(Z_t) (\rho dW_t^1 + \bar{\rho} dW_t^2). \end{aligned}$$

Moreover, we have

$$d\left(2Y_t^{\eta^*} G(Z_t, t)\right) = 2G(Z_t, t) dY_t^{\eta^*} + 2Y_t^{\eta^*} dG(Z_t, t) + 2dG(Z_t, t) dY_t^{\eta^*},$$

so substituting the appropriate dynamics to above equation we get the right hand side of (2.18).  $\square$

**Remark 2.6.** Lemma 2.5 ensures that for fixed initial conditions  $(x_0, y_0, z_0, t_0)$  instead of Markov strategy

$$\pi^* = -2Y_t^{\eta^*} G(Z_t, t) \left[ \frac{\lambda(Z_t)}{\sigma(Z_t)} - \frac{\rho b(Z_t)}{\sigma(Z_t)} \frac{G_z(Z_t, t)}{G(Z_t, t)} \right],$$

we can use

$$\hat{\pi}^* = -\left(X_t^{\pi^*} - x_0 + 2y_0 G(z_0, t_0)\right) \left[ \frac{\lambda(Z_t)}{\sigma(Z_t)} - \frac{\rho b(Z_t)}{\sigma(Z_t)} \frac{G_z(Z_t, t)}{G(Z_t, t)} \right].$$

## 3. SMOOTH SOLUTION TO THE RESULTING EQUATION

To solve equation (2.17) with boundary condition  $G(z, T) = -1$  we will consider the following cases separately:

**Case I:**  $\rho^2 \neq \frac{1}{2}$

In this case the following ansatz is made (see Zariphopoulou [19])

$$G(z, t) = -F^\alpha(z, t), \quad \text{where} \quad F(z, T) = 1,$$

to obtain

$$\begin{aligned} F_t + [a(z) - 2\rho b(z)\lambda(z)] F_z + \frac{1}{2}b^2(z)F_{zz} + \frac{1}{\alpha}\lambda^2(z)F \\ + \left[ \frac{1}{2}(\alpha - 1) - \alpha\bar{\rho}^2 \right] b^2(z)\frac{F_z^2}{F} = 0. \end{aligned}$$

Note that for

$$\alpha = \frac{1}{2\rho^2 - 1},$$

we have

$$(3.1) \quad F_t + [a(z) - 2\rho b(z)\lambda(z)] F_z + \frac{1}{2}b^2(z)F_{zz} + (2\rho^2 - 1)\lambda^2(z)F = 0.$$

**Case II:**  $\rho^2 = \frac{1}{2}$

In this case if we substitute

$$G(z, t) = e^{F(z, t)}, \quad \text{where} \quad F(z, T) = 0,$$

we get

$$(3.2) \quad F_t + [a(z) - \sqrt{2}b(z)\lambda(z)] F_z + \frac{1}{2}b^2(z)F_{zz} + \lambda^2(z) = 0.$$

At the end of this section we give a set of assumptions to ensure existence of unique solution to equation (2.17) with boundary condition  $G(z, T) = -1$  for any  $\rho \in [-1, 1]$ .

**Remark 3.1.** From Theorem 1 of Heath and Schweizer [10] it follows that if  $a$ ,  $b$ ,  $b \cdot \lambda$ ,  $\lambda^2$  are Lipschitz continuous,  $\lambda$  is continuous and bounded and  $b^2 > \epsilon > 0$  then there exist unique smooth solutions  $F_1$  and  $F_2$  to the equations (3.1) and (3.2) respectively, which satisfy the Feynman-Kac representation:

$$\begin{aligned} F_1(z, t) &= \mathbb{E}_{z,t} \left[ \exp \left\{ (2\rho^2 - 1) \int_t^T \lambda^2(\tilde{Z}_s) ds \right\} \right], \\ F_2(z, t) &= \mathbb{E}_{z,t} \left[ \int_t^T \lambda^2(\tilde{Z}_s) ds \right], \end{aligned}$$

where

$$d\tilde{Z}_s = \left[ a(\tilde{Z}_s) - 2\rho b(\tilde{Z}_s)\lambda(\tilde{Z}_s) \right] ds + b(\tilde{Z}_s)dW_s, \quad \tilde{Z}_t = z.$$

Note that, if  $\lambda$  is a bounded function, then  $F_1$  and  $F_2$  are bounded and  $G$  is bounded away from 0 for any  $\rho \in [-1, 1]$ .

**Lemma 3.2.** *Suppose  $a, b, b \cdot \lambda, \lambda^2$  are Lipschitz continuous,  $\lambda$  is continuous and bounded,  $b^2 > \epsilon > 0$  and  $F$  is a bounded solution to equation (3.1) or (3.2). Then the first  $z$ -derivative of  $F$  is bounded.*

*Proof.* To get a bound for  $F_z$  it is sufficient to estimate the Lipschitz constant. First of all, note that for  $z_1, z_2 \in (-\infty, a]$  there exists  $L_a > 0$  such that

$$(3.3) \quad |e^{z_1} - e^{z_2}| \leq L_a |z_1 - z_2|.$$

Secondly, for solution to (3.1), using (3.3) and the fact that  $\lambda^2$  is Lipschitz continuous and bounded we obtain existence of  $L > 0$  that

$$\begin{aligned} |F(z, t) - F(\bar{z}, t)| &\leq L \mathbb{E} \left[ \int_t^T \left| \tilde{Z}_s(z, t) - \tilde{Z}_s(\bar{z}, t) \right| ds \right] \\ &\leq LT \mathbb{E} \left[ \sup_{t \leq s \leq T} \left| \tilde{Z}_s(z, t) - \tilde{Z}_s(\bar{z}, t) \right| \right], \end{aligned}$$

where from notational convenience we wrote  $\mathbb{E}f(\tilde{Z}_s(z, t))$  instead of  $\mathbb{E}_{z,t}f(\tilde{Z}_s)$ . Now, it is well known (Theorem 1.3.16 from Pham [18]) there exists  $C_T > 0$  such that

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} \left| \tilde{Z}_s(z, t) - \tilde{Z}_s(\bar{z}, t) \right| \right] \leq C_T |z - \bar{z}|,$$

which completes the proof in the first case. Naturally, we can get a similar estimate for solution to (3.2).  $\square$

#### 4. GENERAL RESULT AND USEFULL CONCLUSIONS

In this section we formulate our main theorem and compare monotone optimization with mean-variance optimization methods.

**Theorem 4.1.** *Suppose that  $a, b, b \cdot \lambda, \lambda^2$  are Lipschitz continuous,  $\lambda$  is continuous and bounded,  $b$  is bounded and  $b^2 > \epsilon > 0$ . Then there exists a saddle point*

$$(\pi^*(x, y, z, t), \eta^*(x, y, z, t)) \in \mathcal{A}_t \times \mathcal{M}$$

for problem (2.4) such that

$$\begin{aligned} \pi^* &= -2yG(z, t) \left[ \frac{\lambda(z)}{\sigma(z)} - \frac{\rho b(z)}{\sigma(z)} \frac{G_z(z, t)}{G(z, t)} \right], \\ \eta_1^*(\pi^*) &= -\lambda(z), \\ \eta_2^* &= -\bar{\rho} b(z) \frac{G_z(z, t)}{G(z, t)}, \end{aligned}$$

where  $G$  is a unique bounded solution to

$$(4.1) \quad G_t + (a(z) - 2\rho b(z)\lambda(z)) G_z + \frac{1}{2}b^2(z)G_{zz} - \bar{\rho}^2 b^2(z)\frac{G_z^2}{G} + \lambda^2(z)G = 0,$$

with terminal condition  $G(z, T) = -1$ .

*Proof.* It follows from Remark 3.1 and Lemma 3.2 that there exists a unique bounded solution to (4.1), which has bounded derivative  $G_z$ . If we set

$$V(x, y, z, t) := -x + G(z, t)y,$$

then due to (2.12) - (2.16), it is sufficient to prove only that the Markov saddle point  $(\pi^*(x, y, z, t), \eta^*(x, y, z, t))$  belongs to the set  $\mathcal{A}_t \times \mathcal{M}$  and condition (2.11) holds.

Since  $G$  is bounded and  $Y^{\eta^*}$  is a solution to stochastic linear equation with bounded coefficients, then

$$\mathbb{E}_{x,y,z,t}^{\eta} \left[ \sup_{t \leq s \leq T} |G(Z_s, s) Y_s^{\eta^*}| \right] < +\infty \quad \text{for all } \eta \in \mathcal{M}.$$

To prove the same with  $X_t^{\pi^*}$  we use the fact that for fixed initial conditions  $(x_0, y_0, z_0, t_0)$ , strategy  $\pi^*$  might be exchanged with

$$\hat{\pi}^* = - (X_t^{\pi^*} - x_0 + 2y_0 G(z_0, s_0)) \left[ \frac{\lambda(Z_t)}{\sigma(Z_t)} - \frac{\rho b(Z_t)}{\sigma(Z_t)} \frac{G_z(Z_t, t)}{G(Z_t, t)} \right].$$

Now, let us define

$$\zeta(z, t) := - \left[ \frac{\lambda(Z_t)}{\sigma(Z_t)} - \frac{\rho b(Z_t)}{\sigma(Z_t)} \frac{G_z(Z_t, t)}{G(Z_t, t)} \right].$$

Note that  $\zeta \cdot (\mu - r)$  and  $\zeta \cdot \sigma$  are bounded functions since  $\lambda$  and  $\lambda^2$  are bounded. Therefore, the process

$$K_t := X_t^{\pi^*} - x_0 + 2y_0 G(z_0, t_0)$$

is a unique solution to the following equation

$$dK_t = \zeta(Z_t, t)(\mu(Z_t) - r)K_t dt + \zeta(Z_t, t)\sigma(Z_t)K_t dW_t^1.$$

This is a linear stochastic equation with bounded coefficients, which implies that

$$\mathbb{E}_{x_0, y_0, z_0, t_0}^{\eta} \left[ \sup_{t_0 \leq s \leq T} |X_s^{\pi^*}| \right] < +\infty \quad \text{for all } \eta \in \mathcal{M}.$$

It means (2.11) is satisfied and confirms admissibility of  $\pi^*$ .  $\square$

**Relation to mean-variance optimization.** Since motivation of our objective function comes from mean-variance optimization, it is worth to compare our results with solution to the latter. To solve mean-variance optimization problem we consider the following functional

$$\mathcal{I}^{\pi}(x, z, 0) := \mathbb{E}_{x,z,0} X_T^{\pi} - \theta \mathbb{D}_{x,z,0}^2 X_T^{\pi}, \quad \theta > 0,$$

where  $\theta$  is a risk aversion coefficient and

$$\mathbb{D}_{x,z,0}^2 X_T^{\pi} := \mathbb{E}_{x,z,0} [X_T^{\pi} - \mathbb{E}_{x,z,0} X_T^{\pi}]^2.$$

The aim of the investor is to maximize  $\mathcal{I}^\pi(x, z, 0)$  with respect to  $\pi \in \mathcal{A}_0$ .

We use again stochastic control methods to obtain a solution. Namely, we can follow Zhou and Li [22] in noting that

$$\begin{aligned} \sup_{\pi \in \mathcal{A}_0} \mathcal{I}^\pi(x, z, 0) &= \sup_{\pi \in \mathcal{A}_0} \left\{ \mathbb{E}_{x,z,0} X_T^\pi - \theta \mathbb{E}_{x,z,0} [X_T^\pi - \mathbb{E}_{x,z,0} X_T^\pi]^2 \right\} \\ &= \sup_{A \in \mathbb{R}} \sup_{\pi \in \bar{\mathcal{A}}_t} \left\{ A - \theta \mathbb{E}_{x,z,0} (X_T^\pi - A)^2 \right\}, \end{aligned}$$

where

$$\bar{\mathcal{A}}_t = \{ \pi \in \mathcal{A}_t : \mathbb{E}_{x,z,0} X_T^\pi = A, A \in \mathbb{R} \}.$$

In that way we replace unconstrained mean-variance optimization problem with constrained maximization of quadratic objective. Using Lagrange method it is sufficient to minimize functional

$$\begin{aligned} (4.2) \quad I^{\pi(\gamma)}(x, z, t) &:= \mathbb{E}_{x,z,t} \left[ X_T^{\pi(\gamma)} - A \right]^2 - 2\gamma \mathbb{E}_{x,z,t} X_T^{\pi(\gamma)} \\ &= \mathbb{E}_{x,z,t} \left[ X_T^{\pi(\gamma)} - (A + \gamma) \right]^2 - 2A\gamma - \gamma^2, \end{aligned}$$

over a class of admissible controls  $\bar{\mathcal{A}}_t$ , determine the solution  $\pi^*(\gamma)$  and find  $\gamma^*$  such that

$$\mathbb{E}_{x,z,0} X_T^{\pi^*(\gamma^*)} = A.$$

We can use here results from Zawisza [21] where minimization of robust quadratic functional

$$X \rightarrow \sup_{Q \in \mathcal{Q}} \mathbb{E}_{x,z,t}^\eta [X_T^\pi - D]^2$$

was considered (see also Basak and Chabakauri [1] or Laurent and Pham [12]). Namely, if we assume that  $\mathcal{Q} = \{P\}$ , from Theorem 4.1 in Zawisza [21], we have that the optimal strategy for functional (4.2) is given by

$$\pi^*(\gamma, x, z, t) = -(x - (A + \gamma)) \left[ \frac{\lambda(z)}{\sigma(z)} + \frac{\rho b(z)}{\sigma(z)} \frac{H_z(z, t)}{H(z, t)} \right],$$

where  $H$  satisfies

$$H_t + (a(z) - 2\rho b(z)\lambda(z))H_z + \frac{1}{2}b^2(z)H_{zz} - \rho^2 b^2(z) \frac{H_z^2}{H} - \lambda^2(z)H = 0,$$

together with the terminal condition  $H(z, T) = 1$ . Note that  $G = -\frac{1}{H}$  is a solution to

$$G_t + (a(z) - 2\rho b(z)\lambda(z))G_z + \frac{1}{2}b^2(z)G_{zz} - \bar{\rho}^2 b^2(z) \frac{G_z^2}{G} + \lambda^2(z)G = 0,$$

where  $G(z, T) = -1$ . In addition we have

$$(4.3) \quad \pi^*(\gamma, x, z, t) = -(x - (A + \gamma)) \left[ \frac{\lambda(z)}{\sigma(z)} - \frac{\rho b(z)}{\sigma(z)} \frac{G_z(z, t)}{G(z, t)} \right],$$

which shows that the quadratic optimization is consistent with monotone optimization with suitable chosen  $A$  and  $\gamma$  (see Remark 2.6).

Now we find  $\gamma^*$  such that

$$\mathbb{E}_{x,z,0} X_T^{\pi^*(\gamma^*)} = A.$$

Let us define

$$(4.4) \quad P_t := X_t^{\pi^*(\gamma^*)} - (A + \gamma^*).$$

Since  $\zeta \cdot (\mu - r)$  and  $\zeta \cdot \sigma$  are bounded functions,  $P_t$  is solution to stochastic linear equation with bounded coefficients

$$dP_t = \zeta(Z_t, t)(\mu(Z_t) - r)P_t dt + \zeta(Z_t, t)\sigma(Z_t)P_t dW_t.$$

It means that

$$P_t = (x - (A + \gamma^*)) R_t,$$

where

$$R_t := \exp \left\{ \int_0^t \zeta(Z_s, s)(\mu(Z_s) - r) - \frac{1}{2} \zeta^2(Z_s, s) \sigma^2(Z_s) ds + \int_0^t \zeta(Z_s, s) \sigma(Z_s) dW_s \right\}.$$

Using (4.4) we have

$$X_T^{\pi^*(\gamma^*)} = (x - A)R_T + A + \gamma^*(1 - R_T)$$

and then it is easy to see that

$$(4.5) \quad \gamma^*(A) = \frac{(A - x)\mathbb{E}_{z,0} R_T}{1 - \mathbb{E}_{z,0} R_T}, \quad \text{if } \mathbb{E}_{z,0} R_T \neq 1.$$

Finally, note that

$$A - \theta \mathbb{E}_{z,0} \left[ X_T^{\pi^*(\gamma^*)} - A \right]^2 = A - \theta \mathbb{E}_{z,0} [(x - A)R_T + \gamma^*(A)(1 - R_T)]^2,$$

so maximum over  $A$  is attained at

$$(4.6) \quad A^* = x + \frac{1}{2\theta \mathbb{E}_{z,0} \varphi_T^2},$$

where

$$\varphi_T := \frac{R_T - \mathbb{E}_{x,z,0} R_T}{1 - \mathbb{E}_{z,0} R_T}.$$

Substituting it into (4.5) and (4.6) we get

$$A^* = x + \frac{1}{2\theta} \frac{(1 - \mathbb{E}_{z,0} R_T)^2}{\mathbb{D}_{z,0}^2 R_T} \quad \text{and} \quad \gamma^*(A^*) = \frac{1}{2\theta} \frac{(1 - \mathbb{E}_{z,0} R_T) \mathbb{E}_{z,0} R_T}{\mathbb{D}_{z,0}^2 R_T}.$$

Taking into account formula (4.3) we conclude that mean-variance optimal strategy is given by

$$\pi^*(x, z, t) = - \left( x - x_0 - \frac{1}{2\theta} \frac{1 - \mathbb{E}_{z_0,0} R_T}{\mathbb{D}_{z_0,0}^2 R_T} \right) \left[ \frac{\lambda(z)}{\sigma(z)} - \frac{\rho b(z)}{\sigma(z)} \frac{G_z(z, t)}{G(z, t)} \right],$$

whereas monotone optimal strategy looks as follows

$$\pi^*(x, z, t) = - (x - x_0 + 2y_0 G(z_0, 0)) \left[ \frac{\lambda(z)}{\sigma(z)} - \frac{\rho b(z)}{\sigma(z)} \frac{G_z(z, t)}{G(z, t)} \right].$$

We can now summarize our observations.

**Remark 4.2.** Optimization of our modification of monotone mean variance preferences is consistent with classical mean-variance optimization with suitable chosen risk aversion parameter  $\theta$  (which possibly may depend on  $z_0$ ).

**Example.** It is worth to look more closer to the deterministic case in the Black-Scholes model. Namely, assume that

$$\begin{cases} dB_t = rB_t dt, \\ dS_t = \mu S_t dt + \sigma S_t dW_t, \end{cases}$$

where  $r$ ,  $\mu$  and  $\sigma$  are constant. Then monotone strategy is determined by

$$(4.7) \quad \pi^* = - \left( x - x_0 - 2y_0 e^{\lambda^2 T} \right) \frac{\lambda}{\sigma}.$$

To compare it with classical mean-variance case we should first focus on  $R_T$ , which is given by

$$R_T = e^{-\frac{3}{2}\lambda^2 T - \lambda W_T}.$$

This yields

$$\mathbb{E}R_T = e^{-\lambda^2 T}, \quad \mathbb{E}R_T^2 = e^{-\lambda^2 T} \quad \text{and} \quad \mathbb{D}^2 R_T = e^{-\lambda^2 T} - e^{-2\lambda^2 T}.$$

It means that classical mean-variance optimal strategy looks as follows

$$\pi^* = - \left( x - x_0 - \frac{1}{2\theta} e^{\lambda^2 T} \right) \frac{\lambda}{\sigma},$$

and is exactly equal to (4.7) when

$$\theta = \frac{1}{4y_0}.$$

**4.1. Concluding remarks.** In this paper we examined continuous time optimization problem in stochastic factor model assuming that the preference criterion is based on a modification of a monotone mean-variance functional introduced by Maccheroni et al. [13]. We solved the problem under general conditions and showed that the optimal strategy is consistent with a solution to classical mean-variance optimization problem with risk aversion coefficient dependent on stochastic factor. In addition, when there is no observable factor, then the solutions are exactly the same.

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